

A Successive Approximation Algorithm for Computing the Divisor Summatory Function (draft)

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Abstract

An algorithm is presented to compute isolated values of the divisor summatory function in $O(n^{1/3})$ time and $O(\log n)$ space. The algorithm is elementary and uses a geometric approach of successive approximation combined with coordinate transformation.

1 Introduction

Consider the hyperbola from Dirichlet's divisor problem in an xy coordinate system:

$$\begin{aligned} H(x, y) &= xy \\ &= n \end{aligned}$$

The number of lattice points under the hyperbola can be thought of as the number of combinations of positive integers x and y such that their product is less than or equal to n :

$$T(n) = \sum_{x, y: xy \leq n} 1 \quad (1)$$

As such, the hyperbola also represents the divisor summatory function, or the sum of the number of divisors of all numbers less than or equal to n :

$$\tau(x) = \sum_{d|x} 1 = \sum_{x, y: xy = n} 1 \quad (2)$$

$$T(n) = \sum_{x=1}^n \tau(x)$$

One geometric algorithm is to sum columns of lattice points by choosing an axis and solving for the variable of the other axis:

$$T(n) = \sum_{x=1}^n \left\lfloor \frac{n}{x} \right\rfloor \quad (3)$$

which gives an $O(n)$ algorithm. By using the symmetry of the hyperbola (and taking care to avoid double counting) we can do this even more efficiently:

$$T(n) = 2 \sum_{x=1}^{\lfloor \sqrt{n} \rfloor} \left\lfloor \frac{n}{x} \right\rfloor - \lfloor \sqrt{n} \rfloor^2 \quad (4)$$

which gives an $O(n^{1/2})$ algorithm and is in fact the standard method by which the divisor summatory function is computed. Our goal is to break this square-root barrier.

In 1903, Voronoï in [1] made the first significant advance since Dirichlet on the bound on error term for the divisor problem by decomposing the hyperbola into a series of non-overlapping triangles corresponding to tangent lines whose slopes are extended Farey neighbors. We will use a similar approach but where Voronoï produced an exact expression for the error term and estimated its magnitude, we will instead produce an algorithm to determine a precise lattice count for an isolated value of n .

2 Preliminaries

It will be convenient to parameterize the sum in $T(n)$ as:

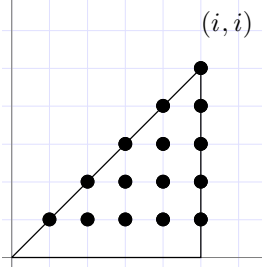
$$S(n, x_1, x_2) = \sum_{x=x_1}^{x_2} \left\lfloor \frac{n}{x} \right\rfloor \quad (5)$$

so that:

$$T(n) = S(n, 1, n) = 2S(n, 1, \lfloor \sqrt{n} \rfloor) - \lfloor \sqrt{n} \rfloor^2 \quad (6)$$

We will also need to count lattice points in triangles. Consider an isosceles right triangle $(0, 0), (i, i), (i, 0)$, i an integer, excluding points on the bottom gives $1 + 2 + \dots + i$ or:

$$\Delta(i) = \frac{i(i+1)}{2}$$



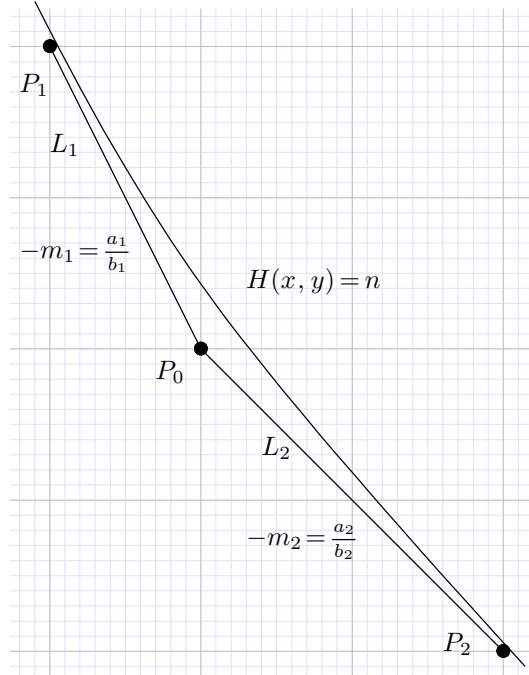
This formula is also applicable to triangles of the form $(0, 0), (i, ai), (i, (a-1)i)$, a a positive integer. If we desire to omit the lattice points on two sides, we can use $\Delta(i-1)$ instead of $\Delta(i)$.

3 Region Processing

Instead of addressing all of the lattice points, let us for the moment consider the sub-task of counting the lattice points in a curvilinear triangular region bounded by two tangent lines and a segment of the hyperbola. If we can approximate the hyperbola by a series of tangent lines, then the area below the lines is a simple polygon and can be calculated directly by decomposing the area into triangles. On the other hand, the region above the two lines can be handled by chopping off another triangle with a third tangent line which creates two smaller curvilinear triangular regions.

We will now go about counting the lattice points in such region. We will do this by first transforming the region into a new coordinate system. This is very simple conceptually but there are a number of details to take care of in order to count lattice points accurately and efficiently. First, the tangent lines are not true tangent lines but are actually shifted to pass through the nearest lattice points. Because of this, tangent lines need to be “broken” on either side of the true tangent point in order to keep them under but close to the hyperbola. Second, the coordinate transformation turns our simple $xy = n$ hyperbola into a general quadratic in two variables. Nevertheless, the recipe at a high level is simply “tangent, tangent, chop, recurse.”

This figure depicts a typical region in the xy coordinate system:



Define two lines L_1 and L_2 whose slopes when negated have positive integral

numerators a_i and denominators b_i :

$$-m_1 = \frac{a_1}{b_1} \quad (7)$$

$$-m_2 = \frac{a_2}{b_2} \quad (8)$$

The slopes are chosen to be Farey neighbors so that the determinant is unity:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2 = 1 \quad (9)$$

and the slopes are rational numbers which we require to be in lowest terms and so we can assume $\gcd(a_1, b_1) = \gcd(a_2, b_2) = 1$.

Assume further that the lines intersect at the lattice point P_0 :

$$(x_0, y_0) \quad (10)$$

with x_0 and y_0 positive integers.

Then the equations for the lines L_1 and L_2 in point-slope form are:

$$\frac{y - y_0}{x - x_0} = -\frac{a_1}{b_1} \quad (11)$$

$$\frac{y - y_0}{x - x_0} = -\frac{a_2}{b_2} \quad (12)$$

and converting to standard form:

$$a_1 x + b_1 y = x_0 a_1 + y_0 b_1 \quad (13)$$

$$a_2 x + b_2 y = x_0 a_2 + y_0 b_2 \quad (14)$$

and defining:

$$c_i = x_0 a_i + y_0 b_i \quad (15)$$

we have:

$$a_1 x + b_1 y = c_1 \quad (16)$$

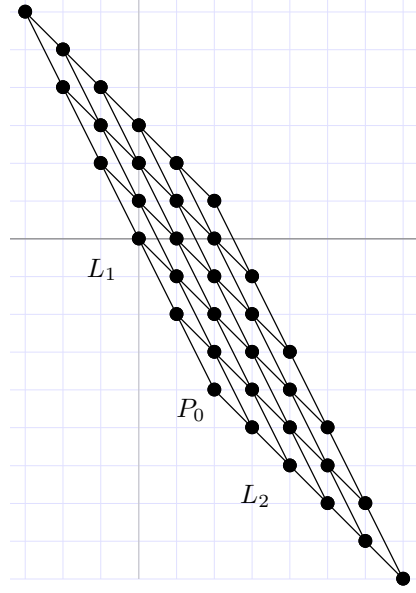
$$a_2 x + b_2 y = c_2 \quad (17)$$

Solving the definitions of c_1 and c_2 for x_0 and y_0 give:

$$x_0 = c_1 b_2 - b_1 c_2 \quad (18)$$

$$y_0 = a_1 c_2 - c_1 a_2 \quad (19)$$

Now observe that the xy lattice points form an alternate lattice relative to lines L_1 and L_2 :



Define a uv coordinate system with an origin of P_0 , L_1 as the v axis and L_2 as the u axis and u and v increasing by one for each lattice point in the direction of the hyperbola. Then the conversion from the uv coordinates to xy coordinates is given by:

$$x = x_0 + b_2 u - b_1 v \quad (20)$$

$$y = y_0 - a_2 u + a_1 v \quad (21)$$

Substituting for x_0 and y_0 and rearranging gives:

$$x = b_2 (u + c_1) - b_1 (v + c_2) \quad (22)$$

$$y = a_1 (v + c_2) - a_2 (u + c_1) \quad (23)$$

Solving these equations for u and v and substituting unity for the determinant provides the inverse conversion from xy coordinates to uv coordinates:

$$u = a_1 x + b_1 y - c_1 \quad (24)$$

$$v = a_2 x + b_2 y - c_2 \quad (25)$$

Because all quantities are integers, equations (22), (23), (24), (25) mean that each xy lattice point corresponds to a uv lattice point and vice versa. As a result, we can choose to count lattice points in either xy coordinates or uv coordinates.

Now we are ready to transform the hyperbola into the uv coordinate system by substituting for x and y in $H(x, y)$ which gives:

$$\begin{aligned} H(u, v) &= (b_2 (u + c_1) - b_1 (v + c_2)) (a_1 (v + c_2) - a_2 (u + c_1)) \\ &= n \end{aligned} \quad (26)$$

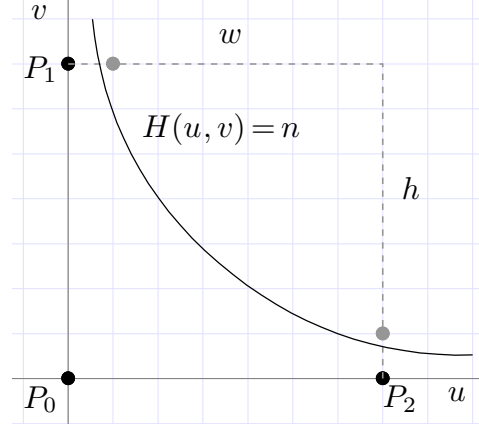
Let us choose a point $P_1 (0, h)$ on the v axis and a point $P_2 (w, 0)$ on the u axis such that:

$$\begin{aligned} H(u_h, h) &= n \\ H(w, v_w) &= n \\ 0 \leq u_h &< 1 \\ 0 \leq v_w &< 1 \\ -dv/du(u_h) &\geq 0 \\ -du/dv(v_w) &\geq 0 \end{aligned}$$

or equivalently that the hyperbola is less than one unit away from the nearest axis at P_1 and P_2 and that the distance to the hyperbola increases as you approach the origin.

With these constraints, the hyperbolic segment has the same basic shape as the full hyperbola: roughly tangent to the axes at the endpoints and strictly decreasing relative to either axis.

This figure depicts a region in the uv coordinate system:



We can now reformulate the number of lattice points in this region R as a function of the eight values that define it:

$$S_R = S_R(w, h, a_1, b_1, c_1, a_2, b_2, c_2) \quad (27)$$

If $H(w, 1) \leq n$, then $v_w \geq 1$ and we can remove the first lattice row:

$$S_R = S_R(w, h - 1, a_1, b_1, c_1, a_2, b_2, c_2 + 1) + w \quad (28)$$

and if $H(1, h) \leq n$, then $u_h \geq 1$ and we can remove the first lattice column:

$$S_R = S_R(w - 1, h, a_1, b_1, c_1 + 1, a_2, b_2, c_2) + h \quad (29)$$

so that the conditions are satisfied.

At this point we could count lattice points in the region bounded by the u and v axes and $u = w$ and $v = h$ using brute force:

$$S_R = \sum_{u,v:H(u,v) \leq n} 1 \quad (30)$$

More efficiently, if we had a formulas for u and v in terms each other, we could sum columns of lattice points:

$$S_W(w) = \sum_{u=1}^w \lfloor V(u) \rfloor \quad (31)$$

$$S_H(h) = \sum_{v=1}^h \lfloor U(v) \rfloor \quad (32)$$

using whichever axis has fewer points, keeping in mind that it could be asymmetric. (Note that these summations are certain not to overcount because by our conditions $V(u) < h$ for $0 < u \leq w$ and $U(v) < w$ for $0 < v \leq h$.)

And so:

$$\begin{aligned} S_R(w, h, a_1, b_1, c_1, a_2, b_2, c_2) &= S_W \\ &= S_H \end{aligned} \quad (33)$$

In fact we can derive formulas for u and v in terms of each other by solving $H(u, v) = n$ (which when expanded is a general quadratic in two variables) for v or u . The resulting explicit formulas for v in terms of u and u in terms of v are:

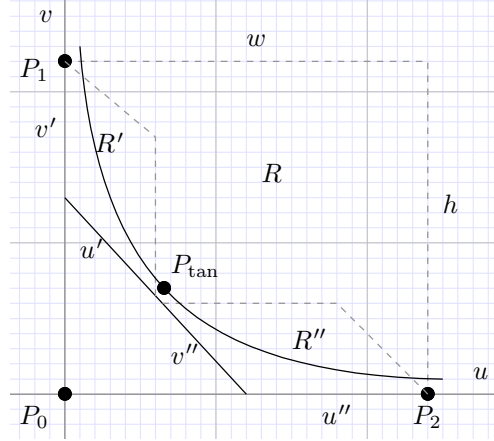
$$V(u) = \frac{(a_1 b_2 + b_1 a_2)(u + c_1) - \sqrt{(u + c_1)^2 - 4 a_1 b_1 n}}{2 a_1 b_1} - c_2 \quad (34)$$

$$U(v) = \frac{(a_1 b_2 + b_1 a_2)(v + c_2) - \sqrt{(v + c_2)^2 - 4 a_2 b_2 n}}{2 a_2 b_2} - c_1 \quad (35)$$

(Note exchanging u for v results in the same formula with subscripts 1 and 2 exchanged.)

As a result we can compute the number of lattice points within the region using a method similar to the method usually used for the hyperbola as a whole. Our goal, however, is to subdivide the region into two smaller regions and process them recursively, only using manual counting at our discretion. To do so we need to remove an isosceles right triangle in the lower-left corner and what will be left are two sub-regions in the upper-left and lower-right.

This figure shows the right triangle and the two sub-regions:



A diagonal with slope -1 in the uv coordinate system has a slope in the xy coordinate system that is the median of the slopes of lines L_1 and L_2 :

$$-m_3 = \frac{a_1 + a_2}{b_1 + b_2} \quad (36)$$

So let us define:

$$a_3 = a_1 + a_2 \quad (37)$$

$$b_3 = b_1 + b_2 \quad (38)$$

Then differentiating $H(u, v) = n$ with respect to u and setting $dv/du = -1$ gives:

$$(a_1 b_2 + b_1 a_2 + 2 a_2 b_2) (u + c_1) = (a_1 b_2 + b_1 a_2 + 2 a_1 b_1) (v + c_2) \quad (39)$$

and the intersection of this line with $H(u, v) = n$ gives the point P_{tan} on the hyperbola where the slope is equal to -1:

$$u_{\text{tan}} = (a_1 b_2 + b_1 a_2 + 2 a_1 b_1) \sqrt{\frac{n}{a_3 b_3}} - c_1 \quad (40)$$

$$v_{\text{tan}} = (a_1 b_2 + b_1 a_2 + 2 a_2 b_2) \sqrt{\frac{n}{a_3 b_3}} - c_2 \quad (41)$$

The equation of a line through this intersection and tangent to the hyperbola is then $u + v = u_{\text{tan}} + v_{\text{tan}}$ which simplifies to:

$$u + v = 2 \sqrt{a_3 b_3 n} - c_1 - c_2 \quad (42)$$

Next we need to find the pair of lattice points $P_4 (u_4, v_4)$ and $P_5 (u_5, v_5)$ such

that:

$$\begin{aligned}
u_4 &> 0 \\
u_5 &= u_4 + 1 \\
-dv/du(u_4) &\geq 1 \\
-dv/du(u_5) &< 1 \\
v_4 &= \lfloor V(u_4) \rfloor \\
v_5 &= \lfloor V(u_5) \rfloor
\end{aligned}$$

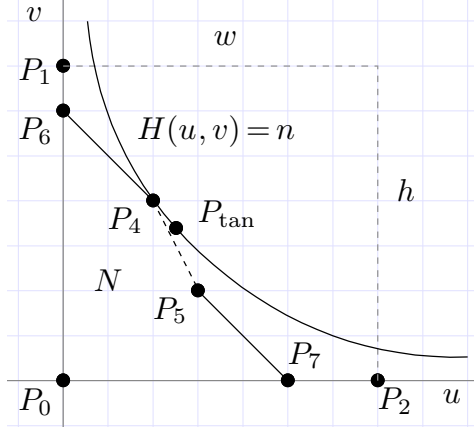
The derivative conditions ensure that the diagonal rays with slope -1 pointing outward from P_4 and P_5 do not intersect the hyperbola. Setting $u_4 = \lfloor u_{\tan} \rfloor$ will satisfy the conditions as long as $u_4 \neq 0$.

Let the point at which the ray from P_4 intersects the v axis be $P_6(0, v_6)$ and the point at which the ray from P_5 intersects the u axis be $P_7(u_7, 0)$. Then:

$$v_6 = u_4 + v_4 \quad (43)$$

$$u_7 = u_5 + v_5 \quad (44)$$

A diagram of all the points defined so far:



Then the number of lattice points above the axes and inside the polygon N defined by points P_0, P_6, P_4, P_5, P_7 is

$$S_N = \Delta(v_6 - 1) - \Delta(v_6 - u_5) + \Delta(u_7 - u_5)$$

or

$$S_N = \begin{cases} \Delta(v_6 - 1) + u_4 & \text{if } v_6 < u_7 \\ \Delta(v_6 - 1) & \text{if } v_6 = u_7 \\ \Delta(u_7 - 1) + v_5 & \text{if } v_6 > u_7 \end{cases} \quad (45)$$

because counting on reverse lattice diagonals starting at the origin we sum $1 + 2 + \dots + (\min(v_6, u_7) - 1)$ plus a partial diagonal if the polygon is not a triangle.

Using the properties of Farey fractions observe that:

$$\begin{aligned} \left| \begin{array}{cc} a_1 & b_1 \\ a_3 & b_3 \end{array} \right| &= a_1(b_1 + b_2) - b_1(a_1 + a_2) = a_1b_2 - b_1a_2 = \left| \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \end{array} \right| = 1 \\ \left| \begin{array}{cc} a_3 & b_3 \\ a_2 & b_2 \end{array} \right| &= (a_1 + a_2)b_2 - (b_1 + b_2)a_2 = a_1b_2 - b_1a_2 = \left| \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \end{array} \right| = 1 \end{aligned}$$

so that m_1 and m_3 are also Farey neighbors and likewise for m_3 and m_2 .

So we can define region R' to be the sub-region with $P'_1 = P_1, P'_0 = P_6, P'_2 = P_4$ and the region R'' to be the sub-region with $P''_1 = P_5, P''_0 = P_7, P''_2 = P_2$ and then the number of lattice points in the entire region is $S_R = S_N + S_{R'} + S_{R''}$ or

$$\begin{aligned} S_R(w, h, a_1, b_1, c_1, a_2, b_2, c_2) &= S_N \\ &+ S_R(u_4, h - v_6, a_1, b_1, c_1, a_3, b_3, c_1 + c_2 + v_6) \\ &+ S_R(w - u_7, v_5, a_3, b_3, c_1 + c_2 + u_7, a_2, b_2, c_2). \end{aligned} \quad (46)$$

This recursive formula for the sum of the lattice points in a region in terms of the lattice points in its sub-regions allows us to use a divide and conquer approach to counting lattice points under the hyperbola.

4 Top Level Processing

Now let us return to the hyperbola as a whole. It should be clear that it is easy in xy coordinates to calculate y in terms of x by solving $H(x, y) = n$ for y :

$$Y(x) = \frac{n}{x} \quad (47)$$

We know that we only need to sum lattice points under the hyperbola up to $\lfloor \sqrt{n} \rfloor$. The point \sqrt{n} is in fact at the $x = y$ axis of symmetry and so the slope at that point is exactly -1 . The next integral slope occurs at -2 , so our first (and largest) region occurs between slopes $-m_1 = 2$ and $-m_2 = 1$. By processing adjacent integral slopes we will start in the middle and work our way back towards the origin.

However, we cannot use the region method for the whole hyperbola because regions become smaller and smaller and eventually a region has a size $w + h \leq 1$. We can find the point where this occurs by taking the second derivative of $Y(x)$ with respect to x and setting it to unity. In other words, the point on the hyperbola where the rate of change in the slope exceeds one per lattice column, which is:

$$x = \sqrt[3]{2n} = 2^{1/3}n^{1/3} \approx 1.26n^{1/3} \quad (48)$$

As a result there is no benefit in region processing the first $O(n^{1/3})$ lattice columns so we resort to the simple method to sum the lattice columns less than

x_{\min} :

$$\begin{aligned} x_{\min} &= C_1 \left\lceil \sqrt[3]{2n} \right\rceil \\ x_{\max} &= \lfloor \sqrt{n} \rfloor \\ y_{\min} &= \lfloor Y(x_{\max}) \rfloor \\ S_1 &= S(n, 1, x_{\min} - 1) \end{aligned} \tag{49}$$

where $C_1 \geq 1$ is a constant to be chosen later.

Next we need to account for the all the points on or below the first line which is a rectangle and a triangle:

$$S_2 = (x_{\max} - x_{\min} + 1) y_{\min} + \Delta(x_{\max} - x_{\min})$$

Because all slopes in this section of the algorithm are whole integers, we have:

$$\begin{aligned} a_i &= -m_i \\ b_i &= 1 \end{aligned}$$

Assume that we have point P_2 and value a_2 from the previous iteration. For the first iteration we will have:

$$\begin{aligned} x_2 &= x_{\max} \\ y_2 &= y_{\min} \\ a_2 &= 1 \end{aligned}$$

For all iterations:

$$a_1 = a_2 + 1$$

The x coordinate of the point on the hyperbola where the slope is equal to m_1 can be found by taking the derivative of $Y(x)$ with respect to x , setting $dy/dx = m_1$, and then solving for x :

$$x_{\tan} = \sqrt{\frac{n}{a_1}} \tag{50}$$

Similar to processing a region (but now in xy coordinates), we now need two lattice points $P_4(x_4, y_4)$ and $P_5(x_5, y_5)$ such that:

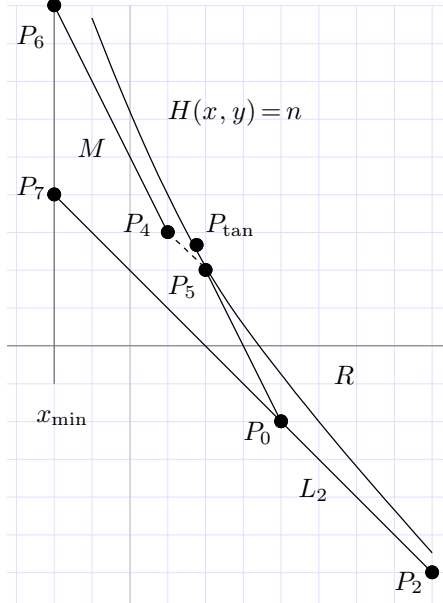
$$\begin{aligned} x_4 &> x_{\min} \\ x_5 &= x_4 + 1 \\ -dy/dx(x_4) &\geq a_1 \\ -dy/dx(x_5) &< a_1 \\ y_4 &= \lfloor Y(x_4) \rfloor \\ y_5 &= \lfloor Y(x_5) \rfloor \end{aligned}$$

To meet these conditions we can set $x_4 = \lfloor x_{\tan} \rfloor$ unless $x_4 \leq x_{\min}$ in which case we can manually count the lattice columns between x_{\min} and x_2 and cease iterating. If so, the remaining columns can be computed as:

$$S_3 = \sum_{x=x_{\min}}^{x_2-1} \left\lfloor \frac{n}{x} \right\rfloor - (a_2(x_2 - x) + y_2)$$

which is the number of lattice points below the hyperbola and above line L_2 over the interval $[x_{\min}, x_2]$.

Now take line L_2 with slope $-a_2$ passing through P_2 , lines L_4 and L_5 with slopes $-a_1$ and passing through P_4 and P_5 and then find the point P_6 where L_4 intersects $x = x_{\min}$ and the point P_0 where L_5 intersects L_2 and the point P_7 where L_2 intersects $x = x_{\min}$ and denote by c_i the y intercept of line L_i .



Now add up the lattice points in the polygon M defined by the points P_0, P_7, P_6, P_4, P_5 but above L_2 by adding the whole triangle corresponding to L_4 , subtracting the portion of it to the right of P_4 , and then adding back the triangle corresponding to L_5 stating at P_5 :

$$S_M = \Delta(c_4 - c_2 - x_{\min}) - \Delta(c_4 - c_2 - x_5) + \Delta(c_5 - c_2 - x_5) \quad (51)$$

where if L_4 is coincident with L_5 , the second two terms cancel each other out.

Then choosing $P_1 = P_5$ (together with P_0 and P_2) and calculating the necessary quantities we have a region R and can now count lattice points using region processing:

$$S_R = S_R(a_1x_2 + y_2 - c_5, a_2x_5 + y_5 - c_2, a_1, 1, c_5, a_2, 1, c_2) \quad (52)$$

so the total sum for this iteration is:

$$S_A(a_1) = S_M + S_R$$

Then we may advance to the next region by setting:

$$\begin{aligned}x'_2 &= x_4 \\y'_2 &= y_4 \\a'_2 &= a_1\end{aligned}$$

Summing all iterations gives

$$S_4 = \sum_{a=2}^{a_{\max}} S_A(a).$$

Finally, the total number of lattice points under the hyperbola from 1 to x_{\max} is

$$S_T = S(1, x_{\max}) = S_1 + S_2 + S_3 + S_4 \quad (53)$$

and therefore the final computation of the divisor summatory function is given by

$$T(n) = 2S_T - \lfloor \sqrt{n} \rfloor.^2 \quad (54)$$

5 Division-Free Counting

Since we calculate S_1 using the traditional method and since the computation will consist entirely of S_1 when $n < 4C_1^6$, it is beneficial to have a faster method of performing this step, albeit by a constant factor. Denote by $l = \lceil \log_2(n) \rceil$ the number of bits needed to represent n . We can avoid an l -bit division in most iterations by using a Bresenham-style calculation (see [2]) and working backwards while computing an estimate of the result of the division based on the previous iteration.

Define $\beta(x) = \lfloor Y(x) \rfloor$, the finite difference $\delta_1(x) = \beta(x) - \beta(x+1)$, and the second-order finite difference $\delta_2(x) = \delta_1(x) - \delta_1(x+1)$. To check whether the value is correct we also need to keep track of the error. So defining the error $\varepsilon(x) = n - x\beta(x) = n - x \lfloor n/x \rfloor = n \bmod x$ gives

$$\begin{aligned}\varepsilon(x) - \varepsilon(x+1) &= (x+a)\beta(x+1) - x\beta(x) \\ &= (x+1)\beta(x+1) - x(\beta(x+1) + \delta_1(x+1) + \delta_2(x)) \\ &= \beta(x+1) - x\delta_1(x+1) - x\delta_2(x)\end{aligned}$$

Introducing the intermediate quantity $\gamma(x) = \beta(x) - (x-1)\delta_1(x)$ and $\hat{\varepsilon}(x)$ as the estimate of the error assuming $\delta_2(x)=0$ then

$$\begin{aligned}\hat{\varepsilon}(x) &= \varepsilon(x+1) + \gamma(x+1) \\ \delta_2(x) &= \left\lfloor \frac{\hat{\varepsilon}(x)}{x} \right\rfloor \\ \delta_1(x) &= \delta_1(x+1) + \delta_2(x) \\ \varepsilon(x) &= \hat{\varepsilon}(x) - x\delta_2(x) \\ \gamma(x) &= \gamma(x+1) + 2\delta_1(x) - x\delta_2(x) \\ \beta(x) &= \beta(x+1) + \delta_1(x).\end{aligned}$$

Over the range $x_1 \leq x \leq x_2$ these integer quantites are bounded in size by $x, \varepsilon(x), |\hat{\varepsilon}(x)| \leq x_2, |\gamma(x)| \leq \max(2n/(x_1^2 + x_1), x_2), \beta(x) \leq n/x_1, \delta_1(x) \leq n/x_1^2 + 1, |\delta_2(x)| \leq 2n/x_1^3 + 2$.

For $\sqrt[3]{2n} < x \leq \sqrt{n}$, $\delta_2(x) \in \{-1, 0, 1, 2\}$ and so

$$\left\lfloor \frac{\hat{\varepsilon}(x)}{x} \right\rfloor = \begin{cases} 2 & \text{if } \hat{\varepsilon}(x) \geq 2x; \\ 1 & \text{if } x \leq \hat{\varepsilon}(x) < 2x; \\ -1 & \text{if } \hat{\varepsilon}(x) < 0; \\ 0 & \text{otherwise;} \end{cases}$$

and thus $\beta(x), \gamma(x), \delta_1(x), \varepsilon(x)$ can be computed from $\beta(x+1), \gamma(x+1), \delta_1(x+1), \varepsilon(x+1)$ using only addition and subtraction of $\frac{1}{2}l$ -bit quantities except $\beta(x)$ which is $\frac{2}{3}l$ bits. Note that $\hat{\varepsilon}(x) \geq 2x$ is very rare over this range and if $\hat{\varepsilon}(x) \geq 3x$, it means that $x < \sqrt[3]{2n}$. For $n^{1/6} \leq x \leq \sqrt[3]{2n}$ we can add the modest division $\lfloor \hat{\varepsilon}(x)/x \rfloor$ between two $\frac{1}{3}l$ -bit values, $\gamma(x)$ and $\delta_1(x)$ grow to $\frac{2}{3}l$ bits and $\beta(x)$ grows to $\frac{5}{6}l$ bits. For $x < n^{1/6}$ we can sum using ordinary division.

6 Algorithms

In this section we present a series of algorithms based on the previous sections. The short-hand notation $F(x)$: expression signifies a functional value that remains unevaluated until referenced.

The first algorithm is a straightforward version of the basic successive approximation method. A literal implementation based on this description will offer many opportunities for optimization. Various formulas have been slightly modified so that the entire algorithm can be implemented using only unsigned multi-precision integer arithmetic. The operations required are addition, subtraction, multiplication, floor division, floor square root, ceiling square root, and ceiling cube root. If any of the root operations are not available, they may be implemented using Newton's method.

Algorithm 1

Inputs: $n \geq 0, C_1 \approx 10, C_2 \approx 10$
 $\Delta(i) : i(i+1)/2$
 $S_1() : \sum_{x=1}^{x < x_{\min}} \lfloor n/x \rfloor$
 $S_2() : (x_{\max} - x_{\min} + 1)y_{\min} + \Delta(x_{\max} - x_{\min})$
 $S_3() : \sum_{x=x_{\min}}^{x < x_2} \lfloor n/x \rfloor - (a_2(x_2 - x) + y_2)$
 $S_M() : \Delta(c_4 - c_2 - x_{\min}) - \Delta(c_4 - c_2 - x_5) + \Delta(c_5 - c_2 - x_5)$
 $x_{\max} \leftarrow \lfloor \sqrt{n} \rfloor, y_{\min} \leftarrow \lfloor n/x_{\max} \rfloor, x_{\min} \leftarrow \min(\lceil C_1 \sqrt[3]{2n} \rceil, x_{\max})$
 $s \leftarrow 0, a_2 \leftarrow 1, x_2 \leftarrow x_{\max}, y_2 \leftarrow y_{\min}, c_2 \leftarrow a_2 x_2 + y_2$
loop
 $a_1 \leftarrow a_2 + 1$
 $x_4 \leftarrow \lfloor \sqrt[n/a_1] \rfloor, y_4 \leftarrow \lfloor n/x_4 \rfloor, c_4 \leftarrow a_1 x_4 + y_4$
 $x_5 \leftarrow x_4 + 1, y_5 \leftarrow \lfloor n/x_5 \rfloor, c_5 \leftarrow a_1 x_5 + y_5$
if $x_4 < x_{\min}$ **then exit loop end if**

```

 $s \leftarrow s + S_M() + S_R(a_1 x_2 + y_2 - c_5, a_2 x_5 + y_5 - c_2, a_1, 1, c_5, a_2, 1, c_2)$ 
 $a_2 \leftarrow a_1, x_2 \leftarrow x_4, y_2 \leftarrow y_4, c_2 \leftarrow c_4$ 
end loop
 $s \leftarrow s + S_1() + S_2() + S_3()$ 
return  $2s - x_{\max}^2$ 
function  $S_R(w, h, a_1, b_1, c_1, a_2, b_2, c_2)$ 
 $\Delta(i) : i(i+1)/2$ 
 $H(u, v) : (b_2(u+c_1) - b_1(v+c_2))(a_1(v+c_2) - a_2(u+c_1))$ 
 $U_{\tan}() : \left\lfloor \sqrt{(a_1 b_2 + b_1 a_2 + 2 a_1 b_1)^2 n / (a_3 b_3)} \right\rfloor - c_1$ 
 $V_{\text{floor}}(u) : \left\lfloor \left( (a_1 b_2 + b_1 a_2)(u+c_1) - \sqrt{(u+c_1)^2 - 4 a_1 b_1 n} \right) / (2a_1 b_1) \right\rfloor - c_2$ 
 $U_{\text{floor}}(v) : \left\lfloor \left( (a_1 b_2 + b_1 a_2)(v+c_2) - \sqrt{(v+c_2)^2 - 4 a_2 b_2 n} \right) / (2a_2 b_2) \right\rfloor - c_2$ 
 $S_W() : \sum_{u=1}^{u < w} V_{\text{floor}}(u)$ 
 $S_H() : \sum_{v=1}^{v < h} U_{\text{floor}}(v)$ 
 $S_N() : \Delta(v_6 - 1) - \Delta(v_6 - u_5) + \Delta(u_7 - u_5)$ 
 $s \leftarrow 0, a_3 \leftarrow a_1 + a_2, b_3 \leftarrow b_1 + b_2$ 
if  $h > 0 \wedge H(w, 1) \leq n$  then  $s \leftarrow s + w, c_2 \leftarrow c_2 + 1, h \leftarrow h - 1$  end if
if  $w > 0 \wedge H(1, h) \leq n$  then  $s \leftarrow s + h, c_1 \leftarrow c_1 + 1, w \leftarrow w - 1$  end if
if  $w \leq C_2$  then return  $s + S_W()$  end if
if  $h \leq C_2$  then return  $s + S_H()$  end if
 $u_4 \leftarrow U_{\tan}(), v_4 \leftarrow V_{\text{floor}}(u_4), u_5 \leftarrow u_4 + 1, v_5 \leftarrow V_{\text{floor}}(u_5)$ 
 $v_6 \leftarrow u_4 + v_4, u_7 \leftarrow u_6 + v_6$ 
 $s \leftarrow s + S_N()$ 
 $s \leftarrow s + S_R(u_4, h - v_6, a_1, b_1, c_1, a_3, b_3, c_1 + c_2 + v_6)$ 
 $s \leftarrow s + S_R(w - u_7, v_5, a_3, b_3, c_1 + c_2 + u_7, a_2, b_2, c_2)$ 
return  $s$ 
end function

```

The next algorithm gives a flavor for the optimizations that are available. It computes the manual summation of a small region over u or v using a handful of additions, one square root and one division per lattice column. A similar technique can be used to compute V_{floor} for the adjacent values u_4 and u_5 . Making this portion of the computation faster favors larger values of C_2 , the cutoff for small regions. An analogy is that this step is faster for small regions in the same way that an insertion sort is faster than a quicksort for small arrays and the break even point can be determined experimentally.

Algorithm 2

```

 $S_W() : S_I(w, c_1, c_2, a_1 b_2 + b_1 a_2, 2a_1 b_1)$ 
 $S_H() : S_I(h, c_2, c_1, a_1 b_2 + b_1 a_2, 2a_2 b_2)$ 
function  $S_I(i_{\max}, p_1, p_2, q, r)$ 
 $s \leftarrow 0, A \leftarrow p_1^2 - 2rn, B \leftarrow p_1 q, C \leftarrow 2p_1 - 1$ 

```

```

for  $i = 1, \dots, i_{\max} - 1$  do
   $C \leftarrow C + 2, A \leftarrow A + C, B \leftarrow B + q$ 
   $s \leftarrow s + \left\lfloor \left( B - \left\lceil \sqrt{A} \right\rceil \right) / r \right\rfloor$ 
end for
return  $s - (i_{\max} - 1) p_2$ 
end function

```

The next algorithm formalizes the steps of the division-free counting method which can be used for the summation S_1 . Whether this is actually faster depends on many things but for example if $n < 2^{94}$, then $\beta, \delta, |\gamma|, |\varepsilon| < 2^{63}$ for $2^{32} < x < 2^{47}$ and if signed 64-bit addition is a single-cycle operation, then a computation of β using this method is about ten cycles vs. say a hundred cycles for a single multi-precision division.

Algorithm 3

```

 $S_1() : S_Q(1, x_{\min} - 1)$ 
function  $S_Q(x_1, x_2)$ 
   $s \leftarrow 0, x \leftarrow x_2, \beta \leftarrow \lfloor n / (x + 1) \rfloor, \varepsilon \leftarrow n \bmod (x + 1), \delta \leftarrow \lfloor n / x \rfloor -$ 
   $\beta, \gamma \leftarrow \beta - x\delta$ 
  while  $x \geq x_1$  do
     $\varepsilon \leftarrow \varepsilon + \gamma$ 
    if  $\varepsilon \geq x$  then
       $\delta \leftarrow \delta + 1, \gamma \leftarrow \gamma - x, \varepsilon \leftarrow \varepsilon - x$ 
      if  $\varepsilon \geq x$  then
         $\delta \leftarrow \delta + 1, \gamma \leftarrow \gamma - x, \varepsilon \leftarrow \varepsilon - x$ 
        if  $\varepsilon \geq x$  then exit while end if
      end if
    else if  $\varepsilon < 0$  then
       $\delta \leftarrow \delta - 1, \gamma \leftarrow \gamma + x, \varepsilon \leftarrow \varepsilon + x$ 
    end if
     $\gamma \leftarrow \gamma + 2\delta, \beta \leftarrow \beta + \delta, s \leftarrow s + \beta, x \leftarrow x - 1$ 
  end while
   $\varepsilon \leftarrow n \bmod (x + 1), \delta \leftarrow \lfloor n / x \rfloor - \beta, \gamma \leftarrow \beta - x\delta$ 
  while  $x \geq x_1$  do
     $\varepsilon \leftarrow \varepsilon + \gamma, \delta_2 \leftarrow \lfloor \varepsilon / x \rfloor, \delta \leftarrow \delta + \delta_2, \varepsilon \leftarrow \varepsilon - x\delta_2$ 
     $\gamma \leftarrow \gamma + 2\delta - x\delta_2, \beta \leftarrow \beta + \delta, s \leftarrow s + \beta, x \leftarrow x - 1$ 
  end while
  while  $x \geq x_1$  do
     $s \leftarrow s + \lfloor n / x \rfloor, x \leftarrow x - 1$ 
  end while
  return  $s$ 
end function

```


7 Time and Space Complexity

Now we present an analysis of the runtime behavior of algorithm.

Theorem 1 *The time complexity of algorithm [6] when computing $T(n)$ is $O(n^{1/3})$ and the space complexity is $O(\log n)$.*

Before we start, we realize that because $x_{\min} = O(n^{1/3})$ and we handle the values of $1 \leq x < x_{\min}$ manually, the algorithm is at best $O(n^{1/3})$. In this section we desire to show that the rest of the computation is at worst $O(n^{1/3})$ so that this lower bound holds for the entire computation.

Our first task is to count and size all the top-level regions. We process one top level region for each integral slope $-a$ from -1 to the slope at x_{\min} . The value for a at each value of x is given by:

$$a = -\frac{d}{dx}Y(x) = \frac{n}{x^2} \quad (55)$$

and:

$$X(a) = \sqrt{\frac{n}{a}} \quad (56)$$

Choosing $C_1 = 1$ so that $x_{\min} = \sqrt[3]{2n}$, then the highest value of a processed is:

$$a_{\max} = \frac{n}{x_{\min}^2} = \frac{n^{1/3}}{2^{2/3}} \quad (57)$$

so there are $O(n^{1/3})$ top level regions.

How big is each top level region? The change in x per unit change in a is dx/da and so:

$$A = -\frac{d}{da}X(a) = \frac{n^{1/2}}{2a^{3/2}} \quad (58)$$

Assume for the moment that the number of total regions visited while processing a region of size A is:

$$N(A) = O(A^G)$$

noting that the cost of processing a region (excluding the cost of processing its sub-regions) is $O(1)$ and so the total number of regions is representative of the total cost.

Now we sum the number of sub-regions processed across all top level region:

$$\begin{aligned} N_{\text{total}} &= \sum_{a=2}^{a_{\max}} N(A) = O\left(\int_1^{a_{\max}} N(A) da\right) \\ &= O\left(\int_1^{a_{\max}} \left(\frac{n^{1/2}}{2a^{3/2}}\right)^G da\right) \end{aligned} \quad (59)$$

We can classify three cases depending on the value of G because the outcome of the integration depends on the final exponent of a :

$$N_{\text{total}} = \begin{cases} O(n^{1/3}) & \text{if } G < 2/3; \\ O(n^{1/3} \log n) & \text{if } G = 2/3; \\ O(n^{G/2}) & \text{if } G > 2/3. \end{cases}$$

(Note that we cannot get below $O(n^{1/3})$ even if $G = 0$ because we have at least $a_{\text{max}} = O(n^{1/3})$ top level regions.)

Now let us analyze the exponent in $N(A)$. In order to determine the number of regions encountered in the course of processing a region of size A , we need to analyze the recursion depth. The recursion will terminate when w or h is unity because by our conditions it is then impossible for the region to contain any more lattice points. Our next task is to measure the size of such a region and so we need to know how many x lattice columns that terminal region represents.

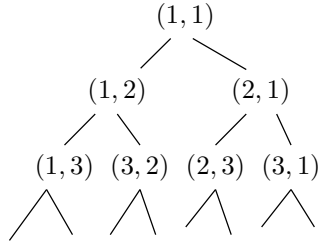
We can use the transformation between uv and xy coordinates given by (20) to compute the difference between the x coordinates of P_2 at $(1, 0)$ and P_1 at $(0, 1)$, assuming the smallest case with $w = h = 1$:

$$\Delta x = x_2 - x_1 + 1 \geq (x_0 + 1 \cdot b_2 - 0 \cdot b_1) - (x_0 + 0 \cdot b_2 - 1 \cdot b_1) + 1 = b_1 + b_2 + 1 > b_1 + b_2 \quad (60)$$

so the size of a terminal region is greater than the sum of the denominators of the slopes of the two lines that define it.

Each time we recurse into two new regions we add a new extended Farey fraction that is the mediant of the two slopes for the outer region. As a result, we perform a partial traversal of a Stern-Brocot tree, doubling the number of nodes at each level. However, for our current purposes we can ignore the numerators because we are interested in the sum of denominators. Because regions cannot overlap, this means that the sum of the denominators at the deepest level of the tree cannot exceed the size of the first region and that only denominators affect the recursion depth.

Next we need to derive a formula for the sum of the denominators of a partial Stern-Brocot tree of depth D . For example, if the first node $(a_1/b_1, a_2/b_2)$ is $(2/1, 1/1)$, the next two nodes are $(2/1, 3/2)$ and $(3/2, 1/1)$. Continuing and ignoring numerators we have the following (b_1, b_2) tree:



At each new level we have twice as many nodes and half of the numbers are duplicated from the previous level and the other half of the numbers are the sum of numbers of their parent node. Since each parent's sum contributes to

exactly two numbers in the children, the sum of the denominators at each level is triple the sum of the previous level. So starting with $1 + 1 = 2$ leads to the sequence $2, 6, 18, 54, \dots$, and denoting by Ω the set of terminal regions, the sum at depth D is therefore

$$A > \sum_{R: R \in \Omega} b_1 + b_2 = 23^D.$$

Because the number of terminal regions is $|\Omega| = 2^D$, we can now place a bound on $|\Omega|$ in terms of A :

$$|\Omega| < \left(\frac{A}{2}\right)^{1/\log_2 3}.$$

Finally, since the total number of regions is $1 + 2 + 4 + \dots + |\Omega| = \sum_{i=1}^D 2^i$, the number of regions as a function of the size A is

$$N(A) = 2|\Omega| - 1 = O\left(A^{1/\log_2 3}\right) \quad (61)$$

and therefore $G = 1/\log_2 3$.

Since $1/\log_2 3 \approx 0.63$, this means that $G < 2/3$ and the proof that the overall time complexity of the algorithm is $O(n^{1/3})$ is complete.

The space complexity is simply our recursion depth which can be at most $O(\log n)$.

8 Higher-Order Divisor Sums

The two-dimensional hyperbola and the functions $\tau(n)$ and $T(n)$ can be generalized to higher dimensions. Using this notation $\tau(n) = \tau_2(n)$ and $T(n) = T_2(n)$. Then the divisor sum $T_3(n)$, the summatory function for $\tau_3(x) = \sum_{abc=x} 1$, can be computed by summing under the three-dimensional hyperbola

$$T_3(n) = \sum_{x,y,z:xyz \leq n} 1 = \sum_{z=1}^n \sum_{x=1}^n \left\lfloor \frac{n}{xz} \right\rfloor = \sum_{z=1}^n T\left(\left\lfloor \frac{n}{z} \right\rfloor\right).$$

Again using the symmetry of this hyperbola we can restrict the outer summation to $\sqrt[3]{n}$ by counting nested “shells”, and avoiding double and triple counting, we

get

$$\begin{aligned}
T_3(n) &= \sum_{z=1}^{\lfloor \sqrt[3]{n} \rfloor} \left[3 \left(2 \sum_{x=z+1}^{\lfloor \sqrt{\frac{n}{z}} \rfloor} \left(\left\lfloor \frac{n}{x} \right\rfloor - z \right) - \left(\left\lfloor \sqrt{\frac{n}{z}} \right\rfloor - z \right)^2 + \left(\left\lfloor \frac{n}{z^2} \right\rfloor - z \right) \right) + 1 \right] \\
&= \sum_{z=1}^{\lfloor \sqrt[3]{n} \rfloor} \left[3 \left(2 \sum_{x=z+1}^{\lfloor \sqrt{\frac{n}{z}} \rfloor} \left\lfloor \frac{n}{x} \right\rfloor - 2z \left(\sqrt{\frac{n}{z}} - z \right) - \left(\left\lfloor \sqrt{\frac{n}{z}} \right\rfloor^2 - 2z \sqrt{\frac{n}{z}} + z^2 \right) + \left\lfloor \frac{n}{z^2} \right\rfloor - z \right) + 1 \right] \\
&= \sum_{z=1}^{\lfloor \sqrt[3]{n} \rfloor} \left[3 \left(2S \left(\left\lfloor \frac{n}{z} \right\rfloor, z+1, \left\lfloor \sqrt{\frac{n}{z}} \right\rfloor \right) - \left\lfloor \sqrt{\frac{n}{z}} \right\rfloor^2 + \left\lfloor \frac{n}{z^2} \right\rfloor + z^2 - z \right) + 1 \right] \\
&= 3 \sum_{z=1}^{\lfloor \sqrt[3]{n} \rfloor} \left(2S \left(\left\lfloor \frac{n}{z} \right\rfloor, z+1, \left\lfloor \sqrt{\frac{n}{z}} \right\rfloor \right) - \left\lfloor \sqrt{\frac{n}{z}} \right\rfloor^2 + \left\lfloor \frac{n}{z^2} \right\rfloor \right) + \lfloor \sqrt[3]{n} \rfloor^3
\end{aligned}$$

where in the last step we use the identity $\sum_{z=1}^k 3(z^2 - z) + 1 = 3(k(k+1)(2k+1)/6 - k(k+1)/2) + k^3$. Since $S(n, x_1, \lfloor \sqrt{n} \rfloor)$ is a partial result in the calculation of $T(n)$, it is also has $O(n^{1/3})$ time complexity when using Algorithm [6]. As a result, we can calculate $T_3(n)$ in

$$\sum_{z=1}^{\lfloor \sqrt[3]{n} \rfloor} O\left(\left\lfloor \frac{n}{z} \right\rfloor^{1/3}\right) = O\left(\int_1^{n^{1/3}} \frac{n^{1/3}}{z^{1/3}} dz\right) = O(n^{5/9}),$$

a modest improvement over $O(n^{2/3})$ using a direct double summation. Similar derivations give $O(n^{2/3})$ for $T_4(n)$ and $O(n^{11/15})$ for $T_5(n)$ or $O(n^{1-4/(3k)})$ for $T_k(n)$ in general.

9 Remarks

It would be possible to simplify the algorithm somewhat by removing the distinction between top level regions and region processing itself by starting with the region defined by $(1/0, 1/1)$. The reason for the current asymmetry is two-fold. First, some of the solutions to the equations are degenerate when $a_i b_i = 0$ and would require special handling anyway. Second, and perhaps more importantly, we can also capitalize on the simpler xy coordinate system where possible.

The two major sections of the algorithm, S_1 and S_4 , are easily parallelizable. The section S_1 can divide summation batches to different processors. The section S_4 can be revised to use a work queue of regions instead of recursion. During region processing, one region can be enqueued and the other processed iteratively. Available processors can dequeue regions that need to be processed.

In fact it turns out that the $S\left(n/z, z + 1 \left\lfloor \sqrt{n/z} \right\rfloor\right)$ terms in the $T_3(n)$ summation skip over the problematic first $O\left((n/z)^{1/3}\right)$ columns by the time z reaches $n^{1/4}$ and then start eroding away the smallest regions as z approaches $n^{1/3}$. Modifying the method slightly and then computing the time complexity of these two portions separately and allowing a_{\max} to decline appropriately we would achieve $O\left(n^{1/2} \log n\right)$ for $T_3(n)$ if we could prove that $G = 1/2$. In any case, using $G = 1/\log_2 3$ at least gives us $O\left(n^{5/9-c+\epsilon}\right)$ for some $c > 0$.

10 Related Work

In [3], Galway presents an improved sieving algorithm that also features region decomposition based on extended Farey fractions as well as coordinate transformation. In [4], applications for the divisor summatory are function presented including computing the parity of $\pi(x)$, the prime counting function, as well as a sketch for a different $O\left(n^{1/3}\right)$ algorithm. In [5], the parity of the prime counting function is studied more closely and several related algorithms are developed.

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